A finite closed regular-faced polyhedral surface with a 5.5.4 vertex

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Abstract

M. Green's *554 problem* asks for a polyhedron whose faces are regular polygons (which "can even be nonconvex and self-intersecting") and which has a vertex surrounded by exactly two pentagons and one square; infinite polyhedra are excluded. We give explicit coordinates in the number field $\mathbb{Q}(\sqrt{1+\sqrt{5}})$ for a finite closed polyhedral surface with this property. It has 92 vertices, 264 edges and 164 faces (130 equilateral triangles, 32 squares and 2 regular pentagons, all of edge length 1); it is connected, orientable, of genus 5, every edge lies in exactly two faces and every vertex link is a single cycle. At the origin exactly one square and two pentagons meet. The realization is self-intersecting and has adjacent coplanar faces (in particular, the square at the 5.5.4 vertex is one quarter of a flat 2×2 square); we state precisely what this means for the different conventions in use. The algebraic and combinatorial claims are proved by hand; the finitely many incidence tests and the counts of intersections and coplanarities are established by a short exact-arithmetic program published with this note, which also re-checks everything else.

1 The problem and the conventions

An *XYZ-acron* is a vertex surrounded, in cyclic order, by an X-gon, a Y-gon and a Z-gon; a polyhedron with at least one such vertex is an *XYZ-acrohedron* [2]. The case 5.5.4 has angle sum $108^\circ + 108^\circ + 90^\circ = 306^\circ$, so such a vertex is locally possible, but no example was known; the catalogue [2] lists it as "none known" and records many near misses [3]. The prize page [1] offers a $200 prize for "the first true 554 or a proof that it's impossible". It states that "the faces can be any regular polygons, and can even be nonconvex and self-intersecting" (that is, star polygons are allowed), and that "infinite polyhedra won't count", since an infinite example (a skew apeirohedron, the 5-5-4 Box Lattice) was found on 16 October 2026. The page does not explicitly address self-intersection of the polyhedron as a whole, coplanar faces, or genus.

Definition 1 formalizes the notion implicit in the acrohedra catalogue to which the prize page links; that catalogue lists minimal solutions with flat vertices (marked Flat), with coplanar adjacent faces (Co-P) and self-intersecting ones (S-Int) [2].

Definition 1. A *finite regular-faced polyhedral surface* is a finite list of points (vertices) in $\mathbb{R}^3$, pairwise distinct, together with a finite list of faces, each a cyclically ordered list of vertices forming a planar convex regular polygon, such that (i) every edge (pair of cyclically consecutive vertices of a face) lies in exactly two faces; (ii) two faces share either nothing, one vertex, or exactly one common edge; (iii) the faces around each vertex form a single cycle (the vertex link is one circle). Conditions (ii) and (iii) are combinatorial. No further condition is imposed on how faces sit in space: faces may intersect each other (self-intersection), including faces that share a vertex.

Conditions (i)–(iii) say that the faces, glued along their common edges, form a closed surface without boundary, without branching edges and without pinched vertices.

Theorem 2. *There is a finite regular-faced polyhedral surface $\mathcal{P}$ in the sense of Definition 1 such that:*

1. *all faces have edge length* 1*; there are* 130 *equilateral triangles,* 32 *squares and* 2 *regular pentagons;*
2. $(V, E, F) = (92, 264, 164)$*; the surface is connected and orientable, with Euler characteristic* -8*, i.e. genus* 5*;*
3. *at the vertex* $O = (0, 0, 0)$ *exactly three faces meet: one square and two regular pentagons, along exactly three edges;*
4. *no vertex lies on an edge or in a face that is not incident to it; no two edges meet except at a common endpoint; distinct coplanar faces never overlap in positive area;*
5. *the open ball of radius* $1/\sqrt{18}$ *around* O *meets only the three faces at* O*.*

2 Five unit vectors

Let $\varphi = (1+\sqrt{5})/2$, $h = \sqrt{\varphi/2} = \frac{1}{2}\sqrt{1+\sqrt{5}}$ and $c = (1-\varphi)/2 = \cos 108^\circ$. Put

$$A = (1, 0, 0), \qquad B = (0, 1, 0), \qquad C = (c, c, h).$$

Then $|A| = |B| = 1$, $|C|^2 = 2c^2 + h^2 = \frac{(1-\varphi)^2}{2} + \frac{\varphi}{2} = 1$ (using $\varphi^2 = \varphi + 1$), $A \cdot B = 0$ and $A \cdot C = B \cdot C = c$. Hence

$$Q_0 = [O, A, A+B, B], \quad P_B = [O, B, \varphi B + C, B + \varphi C, C], \quad P_A = [O, C, A + \varphi C, \varphi A + C, A]$$

are a unit square and two planar regular unit pentagons (a regular unit pentagon $[O, X, \varphi X + Y, X + \varphi Y, Y]$ is spanned by two unit vectors with $X \cdot Y = \cos 108^\circ$). Their angles at O are $90^\circ, 108^\circ, 108^\circ$ and they meet pairwise along OA, OB, OC.

Let $t = (\varphi + 2)/5$, $k = 2(3-\varphi)/5$ and

$$R = t(B + C) + k\,(B \times C), \qquad S = t(A + C) - k\,(C \times A). \tag{1}$$

Lemma 3. $|R| = |S| = 1$, $B \cdot R = R \cdot C = C \cdot S = S \cdot A = \frac{1}{2}$. *Moreover* R *is at distance* 1 *from all five vertices of* P_B*, and* S *from all five vertices of* P_A*.*

Proof. $|B + C|^2 = 2 + 2c = 3 - \varphi$ and $|B \times C|^2 = 1 - c^2 = (\varphi + 2)/4$; the two terms of R are orthogonal, so, using $(3-\varphi)(\varphi+2) = 5$, $|R|^2 = t^2(3-\varphi) + k^2(\varphi+2)/4 = \frac{\varphi+2}{5} + \frac{3-\varphi}{5} = 1$. Next $R \cdot B = R \cdot C = t(1+c) = \frac{\varphi+2}{5} \cdot \frac{3-\varphi}{2} = \frac{1}{2}$, so $|R - B| = |R - C| = |R - O| = 1$. For $U = \varphi B + C$ we have $|U|^2 = \varphi^2 + 1 + 2\varphi c = \varphi + 1$ and $2R \cdot U = (\varphi+1) \cdot 2t(1+c) = \varphi + 1$, so $|R - U|^2 = 1 + (\varphi+1) - (\varphi+1) = 1$; the same holds for $B + \varphi C$. The computation for S is identical with (A, C) in place of (B, C); the signs in (1) are part of the construction (each choice of sign gives one of the two mirror-image apexes, and the program checks the choice made here). □

Thus $P_B \cup \{R\}$ and $P_A \cup \{S\}$ are unit regular pentagonal pyramids, and the cyclic sequence of directions

$$(d_0, d_1, d_2, d_3, d_4) = (A, B, R, C, S) \tag{2}$$

satisfies $d_0 \perp d_1$ and $d_i \cdot d_{i+1} = \frac{1}{2}$ for $i = 1, 2, 3, 4$ (indices mod 5).

3 A closed surface from a five-dimensional box

Let $L = (L_0, \dots, L_4) = (2, 2, 1, 2, 1)$ and consider the box $I = \prod_{i=0}^{4} [0, L_i] \subset \mathbb{R}^5$. Keep a 2-dimensional rectangular face of ∂I exactly when its two varying coordinates are *cyclically consecutive*, $\{i, i+1\}$ (mod 5), the other three coordinates being fixed at 0 or L_m. Subdivide every kept rectangle into unit squares and map the lattice points by the linear map

$$X(q) = q_0 A + q_1 B + q_2 R + q_3 C + q_4 S \in \mathbb{R}^3. \tag{3}$$

A unit square with varying coordinates $(i, i+1)$ and corners q, $q + e_i$, $q + e_i + e_{i+1}$, $q + e_{i+1}$ is mapped to a unit square if $i = 0$ (since $A \perp B$) and to a 60° rhombus if $i = 1, 2, 3, 4$ (since $d_i \cdot d_{i+1} = \frac{1}{2}$); each rhombus is cut along its short diagonal $X(q + e_i)X(q + e_{i+1})$, which has length 1, into two equilateral triangles.

Lemma 4. *The resulting complex $\mathcal{P}_0$ is a closed connected orientable surface of genus* 5 *with* 88 *vertices,* 256 *edges and* 160 *faces (*128 *triangles,* 32 *squares).*

Proof. First take all $L_i = 1$. Then there are $5 \cdot 2^3 = 40$ squares, 80 edges and 32 vertices. An edge in direction i (other four coordinates fixed at endpoints) lies in exactly the two kept squares with coordinate pairs $(i-1, i)$ and $(i, i+1)$. At each vertex the five kept squares are cyclically arranged $e_0e_1, e_1e_2, \dots, e_4e_0$, so the link is a 5-cycle. The edge graph is the 5-cube graph, which is connected. Orient the square with coordinates $(i, i+1)$ and fixed endpoints $\epsilon_m \in \{0, 1\}$ by the ordered pair (e_i, e_{i+1}) times $(-1)^{\sum_{m \neq i, i+1} \epsilon_m}$; along an edge in direction i the two incident squares then induce opposite orientations, so the surface is orientable. (The two squares at an edge in direction i fix different sets of coordinates, namely $i+1$ resp. $i-1$ is fixed; the sign factor exactly compensates this, which is a direct check.) Its Euler characteristic is $32 - 80 + 40 = -8$, so its genus is 5. Stretching intervals and subdividing into unit squares, and cutting squares into triangles, are subdivisions and do not change the surface. For $L = (2, 2, 1, 2, 1)$ one counts 32 corners, 48 midpoints of the $16 + 16 + 16$ edges in the directions $0, 1, 3$, and 8 centres of the $(0, 1)$-rectangles: 88 vertices; there are 32 unit squares from the eight 2×2 $(0, 1)$-rectangles and 16 unit rhombi of each of the four other types, i.e. 64 rhombi and 128 triangles. Then $E = (3 \cdot 128 + 4 \cdot 32)/2 = 256$. □

At O the faces of $\mathcal{P}_0$ are $Q_0, [O, B, R], [O, R, C], [O, C, S], [O, S, A]$.

4 Two local replacements

Put $U_1 = \varphi B + C$, $U_2 = B + \varphi C$, $U_3 = A + \varphi C$, $U_4 = \varphi A + C$. Remove the triangles $[O, B, R], [O, R, C]$ and insert

$$P_B = [O, B, U_1, U_2, C], \quad [B, U_1, R], \quad [U_1, U_2, R], \quad [U_2, C, R];$$

remove $[O, C, S], [O, S, A]$ and insert

$$P_A = [O, C, U_3, U_4, A], \quad [C, U_3, S], \quad [U_3, U_4, S], \quad [U_4, A, S].$$

By Lemma 3 the new triangles are equilateral with edge 1. Each replacement exchanges two adjacent faces of a pentagonal pyramid for the complementary disc (the base and the other three side faces) of the same pyramid; both discs have the same boundary circuit (O, B, R, C, resp. O, C, S, A), so the result $\mathcal{P}$ is again a closed connected orientable surface of the same genus. Each replacement adds 2 vertices, 4 edges and 2 faces, so

$$(V, E, F) = (88, 256, 160) + 2\,(2, 4, 2) = (92, 264, 164),$$

with 130 triangles, 32 squares and 2 pentagons. At O the faces are now exactly Q_0, P_B, P_A, along the three edges OA, OB, OC: a 5.5.4 vertex.

5 Distinctness and non-degeneracy

All coordinates lie in $K = \mathbb{Q}(h)$, where $4h^4 - 2h^2 - 1 = 0$. This polynomial is irreducible: $h^2 = \varphi/2$ and $h \notin \mathbb{Q}(\varphi)$, because the conjugate $(1 - \sqrt{5})/4$ of h^2 is negative, while both embeddings of $\mathbb{Q}(\sqrt{5})$ are real, so a square in $\mathbb{Q}(\sqrt{5})$ has a nonnegative conjugate. Hence $K = \mathbb{Q}(\varphi) \oplus h\,\mathbb{Q}(\varphi)$.

Lemma 5. *A, B, R, C, S are linearly independent over $\mathbb{Q}$. Consequently the 88 points $X(q)$ are pairwise distinct, and $U_1, \dots, U_4$ are distinct from them and from each other.*

Proof. One computes $B \times C = (h, 0, -c)$ and $C \times A = (0, h, -c)$, so the third coordinates of C, R, S are h, $th + \delta$, $th - \delta$ with $\delta = -kc = (3\varphi - 4)/5 \neq 0$. If $aA + bB + rR + qC + sS = 0$ with rational coefficients, the third coordinate gives $h\,(q + t(r + s)) + \delta(r - s) = 0$; the two terms lie in the two summands of K, so $r = s$ and $q + 2tr = 0$; as $t \notin \mathbb{Q}$, $r = s = q = 0$ and then $a = b = 0$. For the extra points: the third coordinate of U_1, U_4 is h and of U_2, U_3 is φh. If $X(q) = U_j$, then $q_2 = q_4 = r$ and $q_3 + 2tr \in \{1, \varphi\}$. In the first case $r = 0$, $q_3 = 1$, and comparing the first two coordinates gives q_0 or q_1 equal to φ, impossible; in the second case comparing the coefficients of $\sqrt{5}$ gives $r = 5/2$, impossible for an integer. The U_j are pairwise distinct by their heights and their A, B components. □

Why three intervals have length 2: with all $L_i = 1$ the points $B + C$ and $A + C$ would be vertices of the surface lying inside the pentagons P_B, P_A. A lattice point with two non-endpoint coordinates is a vertex only when these coordinates are cyclically consecutive; for $L = (2, 2, 1, 2, 1)$ the point $B + C$ has non-endpoint coordinates $\{1, 3\}$ and $A + C$ has $\{0, 3\}$, neither consecutive, so they are not vertices.

Isolation of O. Let $\ell(x, y, z) = x + y + 4z$. Then $\ell(O) = 0$ and $\ell \geq 1$ at every other vertex. Indeed $\ell(A) = \ell(B) = 1$, $\ell(C) = 1 - \varphi + 4h \approx 2.98$ (note $4h > \varphi$), and from (1)

$$\ell(R) = -\tfrac{3}{2} + \tfrac{11\sqrt{5}}{10} + (\tfrac{3}{2} + \tfrac{\sqrt{5}}{10})\sqrt{1 + \sqrt{5}} \approx 4.06, \qquad \ell(S) = \tfrac{5}{2} - \tfrac{13\sqrt{5}}{10} + (\tfrac{1}{2} + \tfrac{3\sqrt{5}}{10})\sqrt{1 + \sqrt{5}} \approx 1.70.$$

Every vertex of $\mathcal{P}_0$ other than O is $X(q)$ with $0 \neq q \in \mathbb{Z}^5_{\geq 0}$, so $\ell \geq 1$ there; and $\ell(U_1) = \ell(U_4) = \varphi + \ell(C)$, $\ell(U_2) = \ell(U_3) = 1 + \varphi\,\ell(C)$, both > 1. (The program checks $\ell \geq 1$ at all 91 vertices exactly.) A face not incident to O is the convex hull of its vertices, so $\ell \geq 1$ on it, while $\ell < 1$ on the open ball of radius $1/|(1, 1, 4)| = 1/\sqrt{18}$ around O. This proves item 5 of Theorem 2: no other part of the surface passes through or near the 5.5.4 vertex.

Items 1–3 of Theorem 2 are proved above; item 4 is a finite list of exact incidence tests (Section 7). This completes the proof of Theorem 2. □

Figure 1 shows the surface and the three faces at O.

6 What the construction is and is not

We list the features that a reader applying a stricter convention should know.

1. **Self-intersection.** The surface is not embedded. There are exactly 72 pairs (edge, face) where an edge crosses the interior of a non-incident face, and no edge meets another edge except at a common endpoint. Accordingly, 83 pairs of faces intersect, each along a segment: 61 pairs of faces with no common vertex, and 22 pairs of faces sharing exactly one vertex, which cross along a segment issuing from that vertex (this happens at 17 vertices, none of which is O, A, B, C or a vertex of P_A, P_B). Three of the 72 edges cross the pentagon P_A, at distances about 0.71, 1.32 and 1.53 from O (well outside the ball of item 5); Q_0 and P_B are crossed by no other face. The prize page allows the *faces* to be self-intersecting (star polygons) and does not explicitly address self-intersection of the polyhedron as a whole; the catalogue it links lists self-intersecting minimal solutions (S-Int).

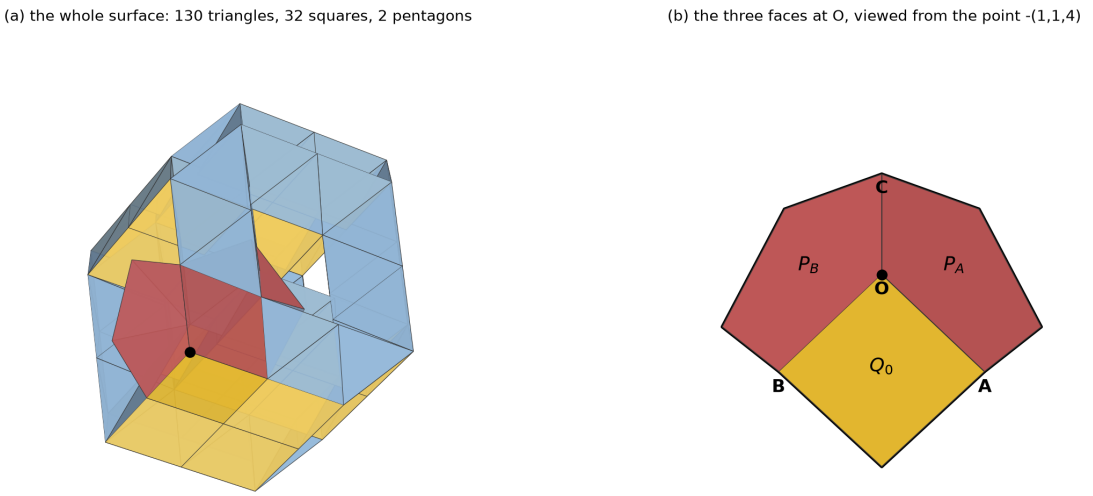


Figure 1: (a) The surface $\mathcal{P}$ (triangles blue, squares yellow, pentagons red; the black dot is O). (b) The three faces Q_0, P_A, P_B at O, viewed from the point $-(1,1,4)$, i.e. from outside the corner.

2. **Genus.** The surface has genus 5, not 0. The prize page states no restriction on genus.

3. **Coplanar adjacent faces.** 124 edges join two coplanar faces (a flat dihedral angle): 92 triangle–triangle and 32 square–square. The maximal flat regions are: 8 squares of side 2 (each made of 4 unit squares; Q_0 is one quarter of such a region), 28 parallelograms with sides $1, 2$ and angles $60°, 120°$ (4 triangles each), 4 isosceles trapezoids with sides $1, 1, 1, 2$ (3 triangles each), 6 single triangles, and the 2 pentagons, which are coplanar with no neighbour. $\mathcal{P}$ also has 8 flat vertices (the centres of the side-2 squares), so in the catalogue's notation it would be marked Flat, Co-P, S-Int. Under the convention of the catalogue, where coplanar adjacent faces are permitted, the faces of $\mathcal{P}$ are the 164 unit polygons listed in Theorem 2. If instead coplanar adjacent faces are merged, O is still surrounded by a regular square (of side 2) and two regular unit pentagons, but A and B become midpoints of edges of that square (the faces no longer meet edge-to-edge), and the parallelogram and trapezoid regions are not regular polygons; so under a convention requiring adjacent faces to be non-coplanar $\mathcal{P}$ would *not* qualify, and we make no claim for that convention.

4. **Finiteness.** $\mathcal{P}$ is a finite surface (92 vertices), not a periodic apeirohedron.

Whether this construction meets the intended terms of the prize is for the prize sponsor to decide; no acceptance is claimed here.

7 Computer verification

The file `acro554_verify.py` (Python 3, standard library only) builds $\mathcal{P}$ directly from formulas (1)–(3) and the replacement rules, and checks, with an assertion for each, every claim of Theorem 2 and every number stated in Section 6. No floating point number enters any decision: each coordinate is stored as $a_0 + a_1 h + a_2 h^2 + a_3 h^3$ with rational a_i, products are reduced by $h^4 = h^2/2 + 1/4$, equality is decided by comparing coefficients (valid since the minimal polynomial has degree 4), and the sign of a nonzero element is decided by rational interval arithmetic

on a rational interval containing h, refined by bisection of $4x^4 - 2x^2 - 1$ (which is increasing on $[\frac{1}{2}, 1]$) until the sign is certain. The program checks:

- the generators (Lemma 3);
- every face: planarity, all 602 squared vertex distances of a regular polygon of edge 1, strict convexity in the given cyclic order;
- pairwise distinctness of the 92 vertices;
- conditions (i)–(iii) of Definition 1, connectedness, a coherent orientation, $\chi = -8$;
- the 5.5.4 incidence at O and $\ell \geq 1$ at all other vertices;
- no vertex on a non-incident closed edge or in a non-incident closed face;
- for all 228 pairs of distinct coplanar faces, a separating edge (no positive-area overlap); the 124 flat edges, the 8 flat vertices, and the shapes of all 48 maximal flat regions (outline corners and side lengths);
- the exact count of the 72 transverse edge–face crossings, the absence of edge–edge contacts, the census of $61 + 22$ intersecting face pairs, and the 3 crossings of P_A.

Run `python3 acro554_verify.py` (about 30 seconds); it also writes `acro554.json` with exact and decimal coordinates and the face list. Run `python3 acro554_verify.py --check acro554.json` to re-verify a stored data file from scratch (including agreement of its decimal coordinates with the exact ones) and compare it with the construction. The construction was in addition checked by an independent program using SymPy polynomial arithmetic modulo $4x^4 - 2x^2 - 1$.

Files

- `acro554_paper.pdf`, `acro554_paper.tex`: this note;
- `acro554_verify.py`: construction and exact verification;
- `acro554.json`: exact coordinates and faces (vertex 0 is O);
- `acro554_figure.png`: the figure.

Use of artificial intelligence

This solution was found with the help of artificial intelligence (AI) tools. AI systems were used to search for the construction, to write and check the proofs, and to write the verification programs; the construction was additionally verified independently with a separate AI-assisted audit. The author has checked the arguments and the computations and takes full responsibility for the content.

References

[1] M. Green, *The 554 Problem*, `https://superliminal.com/geometry/acro/index.html`, accessed 30 October 2026.

[2] J. McNeill, *Acrohedra*, `http://www.orchidpalms.com/polyhedra/acrohedra/acrohedra.html`, accessed 30 October 2026.

[3] J. McNeill, *5-5-4 near misses*, http://www.orchidpalms.com/polyhedra/acrohedra/nearmiss/554.htm, accessed 30 October 2026.