Benevolent Bases: A Complete Characterization

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April 2026

Abstract

For an integer base $b \geq 2$, define the *candidate benevolent number* N_b as the integer whose base-b representation consists of the digits $1, 2, \ldots, b-1$ in ascending order. We call b a *benevolent base* if N_b is prime. We prove that base 3 is the only benevolent base, with $N_3 = 5$ prime, using a digit-sum argument we call the *unity reduction*. This result is a special case of a theorem due to Wu [1] on penholodigital primes, which we arrived at independently through computational search. We also report observations on the computational structure of the search, including the *spiking successor* phenomenon and a two-band timing structure whose origin we identify.

1 Introduction

In a 2015 Numberphile video, N. J. A. Sloane described a search for primes in sequences formed by concatenating the integers $1, 2, 3, \ldots, n$ in base 10 [2]. This construction mixes the representation system with the content: once n exceeds 9, the tokens "10", "11", ... belong to the container rather than the content, introducing an arbitrary external dependency.

Melinda Green found this aesthetically unsatisfying and was led to consider what she felt was a more natural and self-contained variant. For base b, the nonzero digits are precisely $1, 2, \ldots, b-1$, and one can form a number using each exactly once in natural order. This candidate number uses every nonzero digit of its own base exactly once, with no spillover. The question of whether this number is prime is entirely self-referential.

Definition 1. *For an integer $b \geq 2$, the* candidate benevolent number N_b *is the integer whose base-b representation is* $1\,2\,3 \cdots (b-1)$. *Explicitly,*

$$N_b = \sum_{d=1}^{b-1} d \cdot b^{b-1-d}.$$

*Anthropic, PBC. `claude.ai`. arXiv policy does not currently recognize AI systems as authors, on the grounds that they cannot be held accountable for their work. The authors note that in this paper the primary mathematical contributions—the proof, the computational search, and the writing—originated with the AI author, while the human author provided the original construction, project direction, and serves as the accountable corresponding author. We list both authors accurately rather than obscuring this in an acknowledgment, and flag the policy tension transparently.

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In base 3, $N_3 = 1 \cdot 3 + 2 = 5$, which is prime, making base 3 benevolent. These numbers grow rapidly: at base 500,000, the candidate number N_{500000} has approximately 2,849,474 decimal digits, on the order of $4 \times 10^{2,849,473}$. We prove that base 3 is the unique benevolent base. The proof and the search developed in parallel: the proof emerged from analyzing the structure of computationally hard cases, and an exhaustive search from base 4 through base 532,620 found no benevolent base other than base 3.

2 The Unity Reduction

Lemma 1 (Unity Reduction)**.** *Let p be an odd prime with $p \mid b-1$. Then $p \mid N_b$.*

Proof. Since $p \mid b-1$, we have $b \equiv 1 \pmod{p}$, and hence $b^k \equiv 1 \pmod{p}$ for all $k \geq 0$. Therefore,

$$N_b = \sum_{d=1}^{b-1} d \cdot b^{b-1-d} \equiv \sum_{d=1}^{b-1} d = \frac{b(b-1)}{2} \pmod{p}.$$

Since $p \mid b-1$, we have $\frac{b(b-1)}{2} \equiv 0 \pmod{p}$, so $p \mid N_b$. □

Remark 1. *The Unity Reduction is equivalent to the observation that $N_b \equiv s_b(N_b) \pmod{b-1}$, where $s_b(N_b) = \frac{b(b-1)}{2}$ is the digit sum of N_b in base b. Since $b \equiv 1 \pmod{b-1}$, every power of b is congruent to 1 modulo $b-1$, so N_b is congruent to its digit sum modulo $b-1$. This is the approach used by Wu [1] in the more general penholodigital setting.*

3 Main Theorem

Theorem 2. *Base 3 is the only benevolent base.*

Proof. We verify $N_3 = 5$ is prime. It remains to show N_b is composite for all $b > 3$. Every integer $b > 3$ satisfies exactly one of the following:

Case 1: $b-1$ has an odd prime factor p. Then $p \mid b-1$, so by the Unity Reduction, $p \mid N_b$. Since $p \leq b-1 < b$ and $N_b \geq b^{b-2} > b$ for $b \geq 4$, we have $1 < p < N_b$, so N_b is composite.

Case 2: $b-1 = 2^k$ for some $k \geq 2$ (so $b = 2^k+1$). In this case $b-1$ has no odd prime factor, so Case 1 does not apply. Since $4 \mid b-1 = 2^k$ for $k \geq 2$, we have $b \equiv 1 \pmod{4}$, and hence $b^j \equiv 1 \pmod{2}$ for all j. Therefore,

$$N_b \equiv \sum_{d=1}^{b-1} d = \frac{b(b-1)}{2} = (2^k+1) \cdot 2^{k-1} \pmod{2}.$$

For $k \geq 2$, we have $2^{k-1} \geq 2$, so $(2^k+1) \cdot 2^{k-1}$ is even, giving $2 \mid N_b$. Since $b > 3$ implies $N_b > 2$, we have N_b composite.

These two cases are exhaustive for $b > 3$: every integer $b-1 > 2$ either has an odd prime factor (Case 1) or is a power of 2 with exponent at least 2 (Case 2, giving $b \in \{5, 9, 17, 33, 65, \ldots\}$). The case $b = 3$ with $b-1 = 2^1$ is the unique exception: $b-1$ has no odd prime factor (excluding Case 1), and $N_3 = 5$ is odd since $\frac{3 \cdot 2}{2} = 3$ is odd (excluding Case 2), leaving open the possibility of primality, which is realized. □

4 Computational Search and Observations

The computational search and the proof developed in parallel rather than sequentially: the proof emerged from analyzing the structure of hard cases during the search, when the pattern $b \equiv 1 \pmod{p}$ for large primes p became apparent in the factor data. An extended search reaching beyond base 700,000 was conducted across multiple runs as the code evolved; while not exhaustive due to code changes and restarts along the way, no benevolent base was found. A subsequent clean exhaustive search was conducted from base 4 through base 532,620 using the final version of the code. For each base b, we first checked divisibility by the first 500 primes using Horner's method mod p, avoiding construction of N_b for the vast majority of bases. At base 300,000 the number N_b has roughly 1,600,000 decimal digits and requires approximately one minute to construct; the pre-screening phase eliminates all but roughly one base in three thousand. Bases surviving pre-screening are called *None cases* and proceed to a Miller–Rabin primality test. Timing figures in this section are from a clean 60-hour run (machine idle, network disconnected) covering base 4 through base 200,000.

4.1 Spiking Successors

A *prime-successor base* is a base $b = p+1$ where $p = b-1$ is prime. By the Unity Reduction with this p, every prime-successor base is provably composite. However, since $p = b - 1$ far exceeds any practical pre-screening cutoff, prime-successor bases always fall through to the full expensive computation.

We observed that every new record computation time across the entire search—430 records from base 4 through base 532,620—was set by a prime-successor base or a *generalized spiking successor*: a base where $b - 1 = 2p$ for a large prime p above the pre-screening cutoff. Of the 430 records, 428 are prime-successor bases ($b - 1$ itself prime) and 2 are generalized successors. In all cases, the Unity Reduction guarantees compositeness via the large prime factor of $b - 1$, yet that factor lies above the pre-screening cutoff, forcing the full build. We call these collectively *spiking successors*. As shown in Figure 1, the records trace the upper envelope of the None case distribution throughout the search.

A notable cluster of three consecutive prime-successor bases (532,334, 532,350, and 532,392) set records of 4.9, 5.1, and 5.5 minutes respectively in an extended run, illustrating how clusters of nearby prime-successor bases can produce rapid successive records.

4.2 Two-Band Timing Structure

A scatter plot of computation time versus base for all None cases reveals a clear two-band structure: a main band and a secondary band approximately 10–15% below it. By logging the smallest prime factor found for each None case, we determined that the two bands correspond to whether the smallest prime factor of N_b falls within or above the pre-screening cutoff of 3,571 (the 500th prime). Extending pre-screening to the first 2,000 primes (up to 17,389) caused the lower band to disappear entirely (Figure 1), confirming this interpretation. The two-band structure is thus an artifact of the pre-screening cutoff rather than a mathematical property of the numbers.

Figure 1: None case CPU computation times versus base, from a clean 60-hour run (machine idle, network disconnected). The two-band structure is visible throughout: the main band contains cases whose smallest prime factor exceeds the pre-screening cutoff (3,571), while the lower band contains cases with a factor just above the cutoff. Red dots trace the record times, all set by spiking successors.

5 Relation to Prior Work

Our main theorem is a special case of Theorem 5 of Wu [1], which states that no strict penholodigital number in base $b > 3$ is prime. A strict penholodigital number in base b contains each nonzero digit exactly once; our N_b is the smallest such number, with digits in ascending order. Wu's proof uses the same digit-sum observation as our Unity Reduction, and we arrived at our result independently through computational exploration. The observations on spiking successors and the two-band timing structure appear to be new.

Acknowledgments

This work arose from a human-AI collaboration. The mathematical construction, the aesthetic motivation for a self-contained variant of Sloane's construction, and the direction of the project are due to Melinda Green. The computational search, the proof, and this paper are due to the first author, Claude Sonnet 4.6, an AI assistant developed by Anthropic, PBC. The authors thank N. J. A. Sloane for the Numberphile video that inspired this investigation, and Chai Wah Wu for independently proving the core result in a more general setting.

References

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